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by

Donald Erlenkotter

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WESTERN MANAGEMENT SCIENCE INSTITUTE
University of California, Los Angeles

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DONALD ERLENKOTTER

John E. Anderson Graduate School of Management
University of California, Los Angeles

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Abstract

Market area models determine the optimal size of market for a facility. These models are grounded in classical location theory, and express the fundamental trade-off between economies-of-scale from larger facilities and the higher costs of transport to more distant markets. The simpler market area models have been discovered and rediscovered, and applied and reapplied, in a number of different settings. We review the development and use of market area models, and formulate a General Optimal Market Area model that accommodates both economies-of-scale in facilities costs and economies-of-distance in transport costs as well as different market shapes and distance norms. Simple expressions are derived for both optimal market size and optimal average cost, and also the sensitivity of average cost to a non-optimal choice of size. The market area model is used to explore the implications of some recently proposed distance measures and to approximate a large discrete location model, and an extension to price-sensitive demands is provided.

Keywords and phrases

Facilities planning, location, spatial pricing.

1. Introduction

The two most fundamental concepts in the field of spatial analysis for facilities planning are, first, the *location* of the facility, and second, the *size*, or correspondingly the *market area*, of the facility. Many papers have dealt with the issue of facility location; among the several excellent recent surveys of models of this type are those by Francis, McGinnis and White [33]; Hansen et al. [48]; and Hansen, Peeters, and Thisse [49]. Such models typically produce facility sizes and market areas as an integral part of their analysis.

Here we argue that an equally central approach is to focus directly on the issue of facility size and market area under simplified assumptions about the topology and density of the demand space. Models of this type typically assume, perhaps as an approximation, that demand is spread uniformly over an infinite plain, and then derive the optimal facility size and market area, with the question of the determination of precise locations for the facilities left for subsequent analysis. One can protest that this assumption about demand is totally unrealistic, and therefore that such models are of little or no interest. However, an important and emerging counterview is that simple models of this type, which can provide fundamental insights into the nature of the spatial problem, are valuable *complements* to large, complex, and more realistic models [36,46,73]. The simple market area model has characteristics very similar to those of the familiar economic order quantity (EOQ) model, and we believe that it should occupy the same position in the area of spatial analysis that the EOQ model occupies in the field of inventory and production planning [44].

The current position of the market area model in the field of spatial analysis might best be described as obscure, and its origins are less than clear. The three surveys cited above, which include continuous-space models, do not mention it. One recent paper, which uses such a model to plan the sizes and locations of solid waste transfer-stations, refers to the approach taken as the "non-traditional analytical viewpoint" [99, p. 107]. But this approach, which involves the averaging of demand over the corresponding area and then using the size calculated from a market area model to estimate the number of facilities required, dates back at least thirty years [14,15] and has been used in developing countries for at least twenty years [30]. Perhaps the following statement by Newell [73, pp. 359–360] provides an explanation for this obscurity:

> It is not fashionable to communicate the tricks of the trade through the published literature or even in the classroom; they are transmitted via the casual comment, the informal discussion, or the lecture by an artist who can compress into an hour lecture the essence of a book. By the time one finds a use for some trick, one has long since forgotten where he learned it.

Thus these models have been discovered and rediscovered, and applied and reapplied, and each rediscovery and reapplication invariably misses some of the full power of the models.

The applications of market area models go far beyond the basic calculation of facility sizes, which can be quite complex for some service systems [91,96]. In the field of operations research, such models have been used to interpret and guide the use of large integer programming location models [36,37]. In transportation, they have been used to estimate the benefits from transportation improvements [69]. And in economics, they have provided a theoretical base for deriving the form of relationships used in empirical investigations of the costs associated with non-optimal capacities [32,80,81,93,94]. If one extends the market area model to include demand that is elastic with respect to price, prices can be determined jointly with the facility size and market area. Beginning from the model of Mills and Lav [68] and its antecedents, there is an extensive body of literature dealing with spatial pricing policies and equilibrium market areas under various definitions of competition. Surveys of much of this literature have been provided by Beckmann and Thisse [9], Dorward [24], Eaton and Lipsey [26], and Greenhut [40].

In this paper we review the antecedents, development, and occurrences of the market area model, and then present a somewhat more general version of the model encompassing many variations that may be defined by specifying different assumptions about the *shape* of the market, the *distance norm*, the *facility cost* function, and the *transport cost* function. Parameter-invariant and sensitivity analysis properties of the model are examined. We then show how these models may be used to investigate some of the implications of several new distance norms that have been proposed recently, and explore the use of this type of model as an approximation to large discrete location models.

The models examined in the body of the paper will be restricted to those that assume price-inelastic demands and minimize unit costs, for it is this type of model

that seems to have been used in *operational*, as contrasted with *theoretical*, analyses. However, we show in Appendix B how the cost-minimizing model can be related to some of the spatial pricing models provided in the economics literature.

2. The Market Area Model In Perspective

The simplest version of the market area model is based on the following assumptions:

(a) Demand is distributed uniformly over an infinite plain, with density $D > 0$ per unit area.

(b) The cost for a facility producing the amount w is $k + cw$, where $k > 0$ is the fixed cost for the facility and $c \geq 0$ is the variable facility and production cost per unit.

(c) Unit transport costs are proportional to the distance traveled, at the rate $t > 0$ per item per unit distance.

(d) The market area to be served is circular, with radius R, and the facility is to be located at the center of the market.

(e) Distances are measured according to the Euclidean norm.

(f) Subject to assumptions (a) – (e), the average cost per unit of demand (or per unit area) is to be minimized.

Under these assumptions, the facility size w is determined by the area of the market, A:

$$w = AD = \pi DR^2. \tag{2.1}$$

Total costs for the facility are

$$\begin{aligned} & k + cw + \int_0^R t(2\pi rD)r\,dr \\ = \; & k + cw + \tfrac{2}{3}\pi tDR^3 \\ = \; & k + cw + \tfrac{2}{3}t(\pi D)^{-1/2}w^{3/2}. \end{aligned}$$

Average cost per unit of demand is

$$kw^{-1} + c + \tfrac{2}{3}t(\pi D)^{-1/2}w^{1/2}.$$

Since c has no influence on the minimum, we seek to minimize

$$C(w) = kw^{-1} + \tfrac{2}{3}t(\pi D)^{-1/2}w^{1/2}. \tag{2.2}$$

The average unit cost expression $C(w)$ in (2.2) is of the form

$$C(w) = c_1 w^{a_1} + c_2 w^{a_2},$$

and from Appendix A, (A.8) and (A.10), we have the optimal solution values

$$w^* = \left(\frac{3k}{t}\right)^{2/3} (\pi D)^{1/3} \tag{2.3}$$

and

$$C(w^*) = \left(\frac{3t^2k}{\pi D}\right)^{1/3}. \tag{2.4}$$

From (2.1) and (2.3) we have

$$A^* = w^*/D = \pi^{1/3}\left(\frac{3k}{tD}\right)^{2/3} \tag{2.5}$$

and

$$R^* = \left(\frac{3k}{\pi t D}\right)^{1/3}. \tag{2.6}$$

The implications of (2.3) – (2.6) for the sensitivity of the solution to the various parameters are evident. In particular, except for A^* all results are relatively insensitive to the demand density D.

A further sensitivity result expresses the relative impact on average unit cost of a non-optimal choice of w. From (A.11) in Appendix A, we have

$$\frac{C(w)}{C(w^*)} = \frac{1}{3}\left[\left(\frac{w^*}{w}\right) + 2\left(\frac{w}{w^*}\right)^{1/2}\right]. \tag{2.7}$$

The quite remarkable result (2.7), which is independent of the individual cost and demand parameters, is equivalent to the "Cost vs. Number" relationship given by Geoffrion [37].

Another noteworthy characteristic of the optimal cost in (2.4) is that the optimal value of the average transport cost, expressed by the second term in (2.2), is always

exactly twice the optimal value for average fixed cost, given by the first term in (2.2). Again, this property is independent of the cost and demand parameters (see (A.6) in Appendix A). This result seems to have been noted first by Bos [13, p. 33], and has been derived in a more general context by Starrett [84, p. 433].

A precise origin for this simplest market area model seems to be difficult to assign. Assumption (a) has been employed by Lösch [60] and his predecessors, and models of this sort often are called "Löschian." But Lösch was concerned with price-sensitive demands and more general equilibrium properties, and did not give a specific formulation of this type. Two early comprehensive studies on locational analysis, both published in 1956, do not mention such a model [38,52].

Early instances of the expression (2.3) for optimal facility size appear in papers by Bowman [14] and Bowman and Stewart [15]. These authors apparently did not derive the complete formal model. Instead, they hypothesized the form of the relationship in (2.2) and then estimated the coefficients of w^{-1} and $w^{1/2}$ by regression. This, of course, is an eminently reasonable approach when one has data and is uncertain about the precise nature of market shapes, distance norms, etc. The estimated relationship then was minimized to obtain the equivalent of (2.3).

The most comprehensive of the early derivations of this model is that of Bos [13], who considers other market shapes as well. An early derivation from a Polish source is given by Mycielski and Trzeciakowski [72]. An even earlier derivation of optimal dairy product supply areas is given by Olson [75] for a more general facility cost function; Olson also discusses the case of a hexagonal market shape. The problem of finding an optimal supply area, where dispersed raw materials or agricultural products are transported to a central processing facility, is, of course, equivalent to the market area problem if assumptions corresponding to (a) – (f) are made. Applications of this type of supply area model seem to be quite common [1,21,34,53,98].

As a part of more extensive studies, additional derivations of this simple market area model appear in papers by Leamer [58], Newell [73], and Capozza and Van Order [17]. Mohring and Williamson [69] have developed the model with the same more general facility cost function used by Olson [75]. As a colleague has suggested, this model may be one of those for which rederivation is a more efficient strategy than a search through the rather diffuse literature in which the model has appeared. But such rederivations invariably seem to miss some of the solution characteristics

given above.

An obvious shortcoming of the simple model is the restriction to circular market areas, since circles cannot cover an infinite plain without gaps or overlapping. The question of the configuration of the boundary between neighboring centers of production is even older than the market area problem, evidently dating back almost a century and a half [82]. Many authors have explored this topic subsequently. Some of the earlier writings are collected in [22], and more recent work on defining market area boundaries appears in papers by Eaton and Lipsey [28], Hanjoul and Thill [47], Keeney [54], Lowe and Hurter [64], and Von Hohenbalken and West [87]. Boundaries between service facilities have received special attention [19,57]. Of the most recent work in this area, that of Beckmann and Puu on continuous flow systems offers especially provocative insights into market boundaries [6,8,79].

Given the assumptions we have made of uniformity in the distribution of demand and identical costs for facilities, one is led naturally to consideration of space-filling regular polyhedra as market shapes: triangles, squares or diamonds, and hexagons. Regular hexagons, in particular, have been shown to be the most efficient market shapes for the Euclidean distance norm [2,11,12,39,60,86]. However, for other distance measures such as the rectilinear norm, or "Manhattan metric," diamond-shaped market areas are more efficient [34,57]. Triangular market areas seem to be of little interest, and we shall not consider them further.

In the spatial pricing literature, much attention has been devoted to hexagonal market areas with rounded corners, since in some cases delivered prices near the extremities of the market area can be so high that demand is reduced to zero [3,4,23,35,39,41,42,43,50,70,85]. However, this refinement is not relevant to the unit-cost-minimizing models addressed here.

Several authors have examined the modified simple market area model with assumption (d) changed to a hexagonal or square market shape, with the other assumptions remaining unmodified [13,58,73]. Such a change alters only the coefficient of $w^{1/2}$ in (2.2), and the general properties of the model remain the same. The modified versions of the model turn out to be special cases of the more general model presented in the following section, and we shall examine them after that model is developed.

3. The General Optimal Market Area Model

To extend the applicability of the market area model, we present here a more general version of the simple model discussed in the previous section. The intent is to provide a model that is as general as possible *while still retaining the useful properties found for the simple model.* I. e., we desire single-term closed-form expressions for the optimal facility size w^* as in (2.3) and the optimal unit cost $C(w^*)$ as in (2.4), and the same type of sensitivity result as expressed by (2.7). This requires that the average unit cost expression for the general model be of the form

$$C(w) = c_1 w^{a_1} + c_2 w^{a_2},$$

where $a_1 < 0$ and $a_2 > 0$, and $c_1, c_2 > 0$.

To define the General Optimal Market Area (GOMA) model, we modify assumptions (b) – (e) to the following:

(b′) The cost for a facility producing the amount w is $kw^\alpha + cw$, where $k > 0$, $0 \le \alpha < 1$, and $c \ge 0$. Thus $\alpha = 0$ corresponds to the original assumption (b). (As in the simple model, the variable cost c does not affect the solution, and so we shall not consider it further in the body of the paper.)

(c′) Unit transport costs are related to the distance traveled, r, by the expression tr^β, where $t > 0$ and $\beta > 0$. Thus $\beta = 1$ specifies the original assumption (c).

(d′) Various regular two-dimensional market shapes may be specified, with the following letter codes designating those investigated here:

C	circle
H	regular hexagon
S	square
D	diamond (a square rotated 45°).

For each of these shapes, the facility will be located at the center of the market.

(e′) Various distance measures may be designated. These include the following cases of the general ℓ_p norm:

M "Manhattan metric" (ℓ_1) distance
E Euclidean (ℓ_2) distance.

Other norms will be introduced in Section 4.

To specify an instance of the General Optimal Market Area model, we use the form GOMA (α, β, s, d), where α and β are the values of the exponents in (b′) and (c′) respectively, s denotes the market shape code in (d′), and d denotes the distance norm in (e′). Thus GOMA (0, 1, C, E) is the simple market area model examined in Section 2. If the symbol "*" appears in the position for s or d, the property or formula is valid for all cases of s or d.

The isoelastic facilities cost function in (b') is a form well-established by empirical studies for a variety of processes and industries, and can be derived from the physical relationships embodied in many processes [45]. It has been used in a market area model by Mohring and Williamson [69], who derived the GOMA (α, 1, C, E) model (see also [30]). Olson [75] has derived the same model in a supply area context. The isoelastic transport cost function in (c′) with $0 < \beta < 1$ allows for economies-of-distance in transport costs as suggested by Hansen, Peeters, and Thisse [49, pp. 229–230]. Although the model allows $\beta > 1$, it seems unlikely that this case would occur in reality since then a series of linked short trips would be preferred to a single trip for the entire distance covered, and the effective transport cost function would reflect this choice.

For the GOMA (α, β, C, E) model, total costs for the facility are

$$\begin{aligned} & kw^{\alpha} + \int_0^R t(2\pi r D) r^{\beta} dr \\ = \; & kw^{\alpha} + \frac{2\pi}{2+\beta} t D R^{2+\beta} \\ = \; & kw^{\alpha} + \frac{2t}{2+\beta} (\pi D)^{-\beta/2} w^{1+\beta/2} \end{aligned}$$

since $w = \pi D R^2$ from (2.1). Average cost per unit of demand is

$$C(w) = kw^{\alpha-1} + \frac{2t}{2+\beta} (\pi D)^{-\beta/2} w^{\beta/2} \tag{3.1}$$

which, for $\alpha = 0$ and $\beta = 1$, provides (2.2). Observe that (3.1) may be written in the form

$$C(w) = kw^{\alpha-1} + \sigma(\beta, s, d) t \left(\frac{w}{D}\right)^{\beta/2} \tag{3.2}$$

where

$$\sigma(\beta,\ \mathrm{C},\ \mathrm{E}) = \frac{2\pi^{-\beta/2}}{2+\beta}$$

is a "configuration factor" reflecting the market shape s, the distance metric d, and the transport cost exponent β. The form (3.2) expresses the average unit cost for *any* GOMA (α, β, s, d) model with a regular two-dimensional market shape, provided that $\sigma(\beta, s, d)$ can be evaluated. Since w/D defines the area of the market, $\sigma(1, s, d)$ gives the average distance to demand points from the market center for a market of unit area, as is evident from (3.2). We shall return to the evaluation of some specific $\sigma(\beta, s, d)$ after developing the properties of the cost-minimizing solution for (3.2).

From Appendix A, (A.8) and (A.10), we obtain the optimal solution values

$$w^* = \left[\frac{2(1-\alpha)kD^{\beta/2}}{\beta t\sigma(\beta, s, d)}\right]^{\frac{2}{\beta+2(1-\alpha)}} \tag{3.3}$$

and

$$C(w^*) = \left(\frac{\beta+2(1-\alpha)}{2}\right)\left[\left(\frac{2k}{\beta}\right)^{\beta/2}\left(\frac{t\sigma(\beta, s, d)D^{-\beta/2}}{1-\alpha}\right)^{1-\alpha}\right]^{\frac{2}{\beta+2(1-\alpha)}}. \tag{3.4}$$

From (2.1) and (3.3) we have

$$A^* = \left[\frac{2(1-\alpha)kD^{-(1-\alpha)}}{\beta t\sigma(\beta, s, d)}\right]^{\frac{2}{\beta+2(1-\alpha)}}. \tag{3.5}$$

To evaluate the relative impact on average unit cost of a non-optimal choice of w, from (A.11) in Appendix A we have

$$\frac{C(w)}{C(w^*)} = \left(\frac{2}{\beta+2(1-\alpha)}\right)\left[\frac{\beta}{2}\left(\frac{w}{w^*}\right)^{-(1-\alpha)} + (1-\alpha)\left(\frac{w}{w^*}\right)^{\beta/2}\right]. \tag{3.6}$$

The relationship (3.6), which generalizes (2.7) for GOMA (0, 1, C, E), depends *only* on the cost function exponents α and β and is independent of the other cost coefficients, the level of demand, the market shape, and the distance norm. Thus it applies to GOMA $(\alpha,\ \beta,\ *,\ *)$.

From (A.6) in Appendix A, we discover that the ratio of the optimal value of the average transport cost, given by the second term in (3.2), to the optimal value

for average facility cost, defined by the first term in (3.2), is

$$\frac{2(1-\alpha)}{\beta}. \tag{3.7}$$

Again, this result is independent of other cost coefficients, the level of demand, the market shape, and the distance norm, and applies to GOMA (α, β, $*$, $*$). A consequence of (3.7) is that greater degrees of economies-of-scale in facilities (lower α) and economies-of-distance in transport (lower β) *both* lead to a higher share for transport in the total optimal average cost. The ratio (3.7) has been noted by Mohring and Williamson [69] for GOMA (α, 1, C, E) and GOMA (α, 1, D, M). In these cases, the ratio may vary from 2 (for $\alpha = 0$) to 0 (for $\alpha = 1$).

Since the normalized average distances $\sigma(\beta, s, d)$ are equivalent to expected distances for points randomly distributed in a market area, sources that derive expected distances are useful references for constructing various market area models. In particular, expected distances have been derived for many problems involving randomly occurring service demands. Sources containing useful formulae for expected distances include [20,29,56,59,83].

To evaluate the configuration factors $\sigma(\beta, s, d)$ for Euclidean distance with the additional market shapes, we can apply the following formula to calculate total transport costs for n-sided regular polygons, where R is defined to be the distance from the center of the market to the closest point on its perimeter:

$$2ntD\int_0^{\pi/n}\int_0^{R/\cos\theta} r^{1+\beta}\,dr\,d\theta = \frac{2ntD}{2+\beta}\int_0^{\pi/n}\left(\frac{R}{\cos\theta}\right)^{2+\beta} d\theta. \tag{3.8}$$

Here, of course, square and diamond shapes are equivalent. The integral (3.8) can be evaluated only for $\beta = 1$; the resulting values for $\sigma(1, s,$ E$)$ are given in Table 1 for hexagonal ($n = 6$) and square ($n = 4$) market areas.

For rectilinear or "Manhattan metric" distance, total transport costs are calculated for a diamond-shaped market area with sides of length $2R$ by

$$4tD\int_0^{R\sqrt{2}}\int_0^{R\sqrt{2}-u}(u+v)^{\beta}\,dv\,du, \tag{3.9}$$

and for square and circular market areas the upper limits on the integrals are (R, R) and $(R, \sqrt{R^2-u^2})$ respectively. The resulting values for $\sigma(\beta, s,$ M$)$ are given in

TABLE 1

Configuration Factors for Average Transport Costs

Market shape	Distance metric	Configuration factors $\sigma(\beta, s, d)$	$\sigma(1, s, d)$	
C	E	$\left(\frac{2}{2+\beta}\right)\pi^{-\beta/2}$	$\frac{2}{3\sqrt{\pi}}$	$= 0.376$
	M	NA	$\frac{8}{3\pi\sqrt{\pi}}$	$= 0.479$
H	E	NA	$\left(\frac{1}{\sqrt{3}}\right)^{1/2}\left(\frac{4+3\ln 3}{6\sqrt{6}}\right)$	$= 0.377$
	M	see (3.11)	$\left(\frac{2}{3\sqrt{3}}\right)^{3/2}\left(1+\frac{7\sqrt{3}}{12}\right)$	$= 0.480$
D[a]	E	NA	$\frac{\sqrt{2}+\ln(1+\sqrt{2})}{6}$	$= 0.383$
	M	$\left(\frac{2}{2+\beta}\right)2^{-\beta/2}$	$\frac{\sqrt{2}}{3}$	$= 0.471$
S[a]	M	$\frac{2(2-2^{-\beta})}{(1+\beta)(2+\beta)}$	$\frac{1}{2}$	$= 0.500$

[a] For rectilinear distance, diamond-shaped market areas are squares rotated 45° with respect to the principal directions of travel, whereas square market areas are aligned with the travel directions.

Table 1; for a circular market area, the integral (3.9) can be evaluated only for $\beta = 1$.

For a hexagonal market shape with inner radius R and area $2\sqrt{3}R^2$, total transport costs with the Manhattan metric are given by

$$4tD\left\{\int_0^{R/\sqrt{3}}\int_0^{u\sqrt{3}}(u+v)^\beta dvdu + \int_{R/\sqrt{3}}^{2R/\sqrt{3}}\int_0^{2R-u\sqrt{3}}(u+v)^\beta dvdu\right\}$$

$$+4tD\int_0^R\int_0^{u/\sqrt{3}}(u+v)^\beta dvdu$$

$$= \frac{2tDR^{2+\beta}}{\sqrt{3}^{1+\beta}(1+\beta)(2+\beta)}\left[(1+\sqrt{3})^{3+\beta} - 2\sqrt{3}^{1+\beta} - (1+\sqrt{3})2^{2+\beta}\right]. \tag{3.10}$$

As in (3.2), we obtain from (3.10)

$$\sigma(\beta,\ \mathrm{H},\ \mathrm{M}) = \frac{2^{-\beta/2}\left[(1+\sqrt{3})^{3+\beta} - 2\sqrt{3}^{1+\beta} - (1+\sqrt{3})2^{2+\beta}\right]}{3^{1+3\beta/4}(1+\beta)(2+\beta)}. \tag{3.11}$$

With the information in Table 1, we now may make some inferences about the sensitivity of optimal decisions and costs to market shape and distance metric, at least for the case of $\alpha = 0$ and $\beta = 1$. Observe that the choice of metric has a more significant effect than the choice of market shape: for the most efficient combinations, σ(1, D, M) is about 25% higher than σ(1, C, E). For the Euclidean metric, there is less than a 2% difference between σ(1, C, E) and σ(1, D, E); for the Manhattan metric, the difference is approximately the same between σ(1, D, M) and σ(1, H, M). The consequences of these differences for the optimal facility size calculated via (3.3) are illustrated in Table 2, where the last column gives the proportionality constant

$$\left[\frac{2(1-\alpha)}{\beta\sigma(\beta,s,d)}\right]^{\frac{2}{\beta+2(1-\alpha)}} \tag{3.12}$$

in (3.3). The largest error in w^* that one might obtain by an incorrect choice of market shape and norm is about 20%. Applying either formula (2.7) or (3.6), this degree of error implies an increase in unit cost of less than 1%. Although this may seem small, the increase in unit cost implied by an incorrect choice of market shape given the correct norm is truly imperceptible.

TABLE 2

Proportionality Constants for Optimal Facility Sizes
($\alpha = 0, \ \beta = 1$)

Market shape	Distance metric	Proportionality constant (3.12)		
C	E	$(9\pi)^{1/3}$	=	3.046
	M	$\pi(\frac{3}{4})^{2/3}$	=	2.593
H	E	$2\sqrt{3}\left(\frac{6}{4+3\ln 3}\right)^{2/3}$	=	3.041
	M	$6\sqrt{3}\left(\frac{3}{12+7\sqrt{3}}\right)^{2/3}$	=	2.589
D[a]	E	$\left(\frac{12}{\sqrt{2}+\ln(1+\sqrt{2})}\right)^{2/3}$	=	3.012
	M	$(3\sqrt{2})^{2/3}$	=	2.621
S[a]	M	$(4)^{2/3}$	=	2.520

[a] For rectilinear distance, diamond-shaped market areas are squares rotated 45° with respect to the principal directions of travel, whereas square market areas are aligned with the travel directions.

4. Exploring New Distance Norms

Our analysis thus far has been restricted to two standard distance measures: the Euclidean and rectilinear norms. Several additional distance measures have been proposed recently for use in location models; research in this area includes both the development of new forms of distance norms [77,89,90] and the empirical fitting of distance functions to data [10,55,61,62,63,92]. We now shall explore how some of these norms may be incorporated into the General Optimal Market Area model, and how that model may be used to evaluate the likely impact of selecting such a new norm. The discussion in the previous section suggests that changing the norm probably will provide a rather modest improvement in unit costs, of one percent or less. However, if this improvement costs little and the new norm is accurate and convenient to use, its selection could be desirable.

The particular norms we shall examine are cases of the "block" norm developed by Ward and Wendell [90]. Block, or polyhedral, norms are distance measures that have polygonal contours. The rectilinear, or Manhattan, metric is one example of a block norm. By using other block norms, we can allow travel in angular directions different from the strictly right-angle ones permitted by the Manhattan metric.

The first case that we shall consider is the diamond-shaped market area with travel allowed along 45° diagonals in addition to the horizontal and vertical directions permitted by the Manhattan metric. This corresponds to the "weighted one-infinity" norm of Ward and Wendell [89]. We shall designate this metric as "A45," since directions are differentiated by integer multiples of 45° angles. If (u, v) are the coordinates of a point in the market area, measured from the center, where $u \geq v \geq 0$, the distance given by the A45 metric is $\rho_1 u + \rho_2 v$, with $\rho_1 = 1$ and $\rho_2 = \sqrt{2} - 1$. This implies that a shortest path is taken from the center to point (u, v) following the directions permitted. Total transport costs for GOMA (α, β, D, A45) with sides of length $2R$ are given by

$$8tD\left\{\int_0^{R/\sqrt{2}}\int_0^{u}(\rho_1 u + \rho_2 v)^\beta dv du + \int_{R/\sqrt{2}}^{R\sqrt{2}}\int_0^{R\sqrt{2}-u}(\rho_1 u + \rho_2 v)^\beta dv du\right\}$$

$$= \frac{2^{2-\beta/2} t D R^{2+\beta}}{\rho_2(1+\beta)(2+\beta)}\left\{(\rho_1+\rho_2)^{1+\beta} - 2(2\rho_1)^{1+\beta} + \frac{(2\rho_1)^{2+\beta} - (\rho_1+\rho_2)^{2+\beta}}{\rho_1 - \rho_2}\right\}. \quad (4.1)$$

If the limit were taken in (4.1) as $\rho_1 \to \rho_2 = 1$, we would obtain the transport costs for the Manhattan metric. For the A45 metric, with $\rho_1 = 1$ and $\rho_2 = \sqrt{2} - 1$, we obtain the following configuration factor $\sigma(\beta, \mathrm{D}, \mathrm{A45})$ for use in (3.2) – (3.5):

$$\sigma(\beta, \mathrm{D}, \mathrm{A45}) = \frac{2^{1-\beta/2}(2 - 2^{(1-\beta)/2})}{(1+\beta)(2+\beta)(2-\sqrt{2})}. \tag{4.2}$$

For comparison with the results in Table 1, the average distance $\sigma(1, \mathrm{D}, \mathrm{A45})$ is 0.402.

The second case we develop is for a hexagonal market area with travel permitted along directions that parallel the three principal axes of the hexagon; i.e., directions that intersect at angles of 60°. This metric will be called "A60." For coordinates (u, v), $u \geq v \geq 0$, measured from the center with a horizontal axis corresponding to a principal axis of the hexagon and closest to that principal axis, the distance from the center for the A60 metric is $\rho_1 u + \rho_2 v$, with $\rho_1 = 1$ and $\rho_2 = 1/\sqrt{3}$. Again, a *shortest path along the allowable directions is implied*. For a hexagon with inner radius R and area $2\sqrt{3}R^2$, total transport costs for GOMA (α, β, H, A60) are calculated according to

$$12tD\left\{\int_0^{\sqrt{3}R/2}\int_0^{u/\sqrt{3}}(\rho_1 u + \rho_2 v)^\beta dv du + \int_{\sqrt{3}R/2}^{2R/\sqrt{3}}\int_0^{2R-u\sqrt{3}}(\rho_1 u + \rho_2 v)^\beta dv du\right\}$$

$$= \left(\frac{2}{\sqrt{3}}\right)^\beta \left(\frac{4\sqrt{3}tDR^{2+\beta}}{2+\beta}\right) \tag{4.3}$$

for $\rho_1 = 1$, $\rho_2 = 1/\sqrt{3}$. From (4.3), the configuration factor $\sigma(\beta, \mathrm{H}, \mathrm{A60})$ as in (3.2) is

$$\sigma(\beta, \mathrm{H}, \mathrm{A60}) = \frac{2^{1+\beta/2}3^{-3\beta/4}}{2+\beta}. \tag{4.4}$$

The average distance $\sigma(1, \mathrm{H}, \mathrm{A60})$ is 0.414.

The last case we shall examine is a hexagonal market area with travel permitted along directions that parallel the three principal axes of the hexagon plus intermediate directions that intersect those directions at angles of 30°. This will be called the "A30" metric, and the weights ρ_1, ρ_2 that give the distance $\rho_1 u + \rho_2 v$ from the center to a point (u, v), $u \geq \sqrt{3}v \geq 0$, measured with respect to a horizontal axis aligned with a permissible direction, are $\rho_1 = 1$ and $\rho_2 = 2 - \sqrt{3}$. For an inner radius of R, total transport costs for GOMA (α, β, H, A30) are

$$12tD\int_0^R\int_0^{u/\sqrt{3}}(\rho_1 u + \rho_2 v)^\beta dv du = \frac{12tDR^{2+\beta}}{(2-\sqrt{3})(1+\beta)(2+\beta)}\left[\left(\frac{2}{\sqrt{3}}\right)^{1+\beta} - 1\right] \tag{4.5}$$

for $\rho_1 = 1$, $\rho_2 = 2 - \sqrt{3}$. From (4.5) we have

$$\sigma(\beta,\ \mathrm{H},\ \mathrm{A30}) = \frac{6(2-\sqrt{3})^{-\beta/2}}{(2\sqrt{3}-3)(1+\beta)(2+\beta)}\left[\left(\frac{2}{\sqrt{3}}\right)^{1+\beta} - 1\right]. \qquad (4.6)$$

Here the average distance $\sigma(1,\ \mathrm{H},\ \mathrm{A30})$ is 0.386, which is close to the average distances with the Euclidean norm given in Table 1.

To help in assessing the implications of these new norms for facility sizes and costs, the proportionality constants (3.12) are given in Table 3 for the three cases represented by (4.2), (4.4), and (4.6), with $\alpha = 0$ and $\beta = 1$. In comparison with the values in Table 2, facility sizes are larger than those obtained with the Manhattan metric and closer to those corresponding to the Euclidean metric. For an example of the cost implications, the difference in facility sizes between GOMA $(0, 1, \mathrm{D}, \mathrm{M})$ and GOMA $(0, 1, \mathrm{D}, \mathrm{A45})$ is about 10%. From (2.7) or (3.6), this would imply an increase in optimal costs of about 0.3% if the solution from the wrong model were used. Such an estimate provides some guidance as to the amount of effort that might be justified in determining an appropriate distance norm. In some applications, estimating distances and costs is complicated by vehicle routing considerations, and obtaining accurate estimates may be quite difficult [20,66,92].

The three norm-shape combinations developed here are intermediate between those with Euclidean and Manhattan metrics, and thus may be useful compromise choices if neither of those metrics seems appropriate. They also have the advantage of providing transport cost expressions that are simple to calculate for $\beta \neq 1$, whereas in some instances the Euclidean and Manhattan metrics are intractable. Convenience is important. For example, GOMA $(\alpha, \beta, \mathrm{H}, \mathrm{A60})$ has the useful property that all points on the perimeter of the hexagon are equidistant from the center. Market area models used to analyze spatial pricing become quite awkward if points on the market area perimeter are *not* equidistant from the center. Economists, who seem to have had a simultaneous fixation on hexagonal market areas and Euclidean distance, might have avoided a lengthy controversy over "rounded" hexagons by a theoretically non-significant shift from the Euclidean to the A60 metric.

5. Approximating Discrete Location Models

An important use of market area models is as an approximation to large discrete location problems, which often are represented by integer programming models.

This idea has been developed fully by Geoffrion [36,37], who emphasizes the insights that can be obtained from the simpler models and the value of using the two types of models in combination. Similar views of the *complementary* relationship between continuous and discrete spatial models have been expressed by Beckmann and Puu [8, pp. 254–255] and Hall [46].

Here we provide another example of the correspondence of results produced by the market area model to those from an integer programming model. The integer programming results are for the 100×100 problem in [31] which has 100 demand and potential facility locations. Each location has one unit of demand, and distances between all locations are Euclidean. Since total demand is 100, this suggests calculating a facility size with the market area model and then dividing that size into 100 to estimate the number of facilities required.

There is no reason here to select a market area model more complicated than the simplest one, GOMA (0, 1, C, E), and so we use (2.3) to calculate the facility size. The estimated number of facilities is provided by

$$\hat{N} = \frac{100}{w^*} = Kk^{-2/3}, \tag{5.1}$$

where K is a constant that depends on t and D, parameters that we shall not vary here. The relationship (5.1) is essentially the same as that used by Geoffrion [36,37] and Leamer [58]; it hypothesizes that doubling the fixed charge k will lead to a 37% decrease in the number of facilities.

There are many ways to estimate the constant K; we shall use a break-even approach that recognizes the continuous nature of $\hat{N}$ in (5.1). From Erlenkotter [31, p. 1007], we have a solution with four facilities at $k = 8000$ having total transportation costs of $55,889$; and a solution with five facilities at $k = 7000$ with transportation costs of $48,720$. For indifference between these two solutions, the breakeven fixed charge k_b is determined by

$$48,720 + 5k_b = 55,889 + 4k_b$$

which yields $k_b = 7169$. This value of k_b corresponds to an estimate of $\hat{N} = 4.5$ in (5.1), and thus (5.1) becomes

$$\hat{N} = 1673k^{-2/3}. \tag{5.2}$$

We now use (5.2) to estimate the number of facilities required, and compare the estimates in Table 4 with the actual integer programming results. If the estimates

TABLE 3

Proportionality Constants for Optimal Facility Sizes
$(\alpha = 0, \quad \beta = 1)$

Market shape	Distance metric	Proportionality constant (3.12)		
D	A45	$[12(\sqrt{2}-1)]^{2/3}$	=	2.913
H	A60	$2^{-1/3}3^{7/6}$	=	2.860
H	A30	$2(2-\sqrt{3})^{2/3}3^{7/6}$	=	2.995

TABLE 4

Estimated Number of Facilities for (100×100) Location Problem

Fixed cost, k	Estimated number of facilities, $\hat{N}$	Optimal number of facilities, N^*
1000	16.73	17
2000	10.54	12
3000	8.04	7
4000	6.64	7
5000	5.72	6
6000	5.07	5
7000	4.57	5
8000	4.18	4

$\hat{N}$ are rounded to the nearest integer, we see that the correspondence is amazingly close, differing by a single facility in just two cases: $k = 2000$ and $k = 3000$. The fit would be even closer if the relationship (5.1) were fitted to average, rather than extreme, data values.

6. Conclusion

The market area model presented here is important to spatial analysis. Much as the familiar EOQ model expresses the fundamental tradeoff between economies from larger orders and increases in inventory holding costs, the market area model portrays the equally fundamental tradeoff between economies-of-scale from larger facilities and the higher costs of transport to more distant markets. As we have shown, it is a model that has been rediscovered and rederived a number of times. In view of its importance and long history, the apparent obscurity of this model seems quite puzzling.

The more general form of the market area model given here encompasses both economies-of-scale in facility costs and economies-of-distance in transport costs. A variety of market shape and distance norm combinations are accommodated within the general model. Even in this extended form, the GOMA model preserves all the useful properties of the simplest model: compact closed-form expressions for the optimal facility size and the optimal unit average cost, and convenient sensitivity relationships for the impact on costs of non-optimal decisions.

We have used the model to explore the implications of some new distance measures and to approximate a large integer programming location model. Such a small, convenient model seems an ideal choice as a means for performing the kinds of rough cost-benefit analysis needed to manage more complex locational modeling exercises. As Berens and Körling [10] have pointed out, even the selection of a distance measure is subject to such cost-benefit considerations. We believe that much remains to be discovered about the potential for this simple model as a part of the overall modeling process.

Appendix A. General Solution Derivation

Consider the following two-term cost function:

$$C(w) = c_1 w^{a_1} + c_2 w^{a_2}, \tag{A.1}$$

where $a_1 < 0$ and $a_2 > 0$, and $c_1, c_2 > 0$. We may apply the most elementary concepts from geometric programming, as developed in Wilde and Beightler [97, pp. 28–30], to derive the cost-minimizing solution w^* to (A.1) and also several useful properties of this solution.

The proportion for each cost term in (A.1) in the optimal solution is given by

$$\frac{c_j {w^*}^{a_j}}{C(w^*)} = v_j^*, \qquad j = 1, 2. \tag{A.2}$$

Hence

$$v_1^* + v_2^* = 1. \tag{A.3}$$

Differentiating in (A.1), we have the necessary condition

$$\frac{dC(w^*)}{dw} = 0 = a_1 c_1 {w^*}^{a_1 - 1} + a_2 c_2 {w^*}^{a_2 - 1}. \tag{A.4}$$

Substituting $c_j {w^*}^{a_j} = v_j^* C(w^*)$ from (A.2) into (A.4), we have

$$\frac{C(w^*)}{w^*} [a_1 v_1^* + a_2 v_2^*] = 0$$

which implies that

$$a_1 v_1^* + a_2 v_2^* = 0. \tag{A.5}$$

Solving (A.3) and (A.5) for the v_j^* yields

$$v_1^* = \frac{a_2}{a_2 - a_1}; \qquad v_2^* = \frac{-a_1}{a_2 - a_1}. \tag{A.6}$$

Hence the proportions of the two cost terms in (A.1) in the optimal solution depend only upon the exponents a_j, and not on the c_j.

To calculate the optimal costs, note that (A.3) implies

$$C(w^*) = C(w^*)^{v_1^*} C(w^*)^{v_2^*}. \tag{A.7}$$

Substituting into (A.7) from (A.2) gives

$$\begin{aligned} C(w^*) &= \left[\frac{c_1 w^{*a_1}}{v_1^*}\right]^{v_1^*} \left[\frac{c_2 w^{*a_2}}{v_2^*}\right]^{v_2^*} \\ &= \left[\frac{c_1}{v_1^*}\right]^{v_1^*} \left[\frac{c_2}{v_2^*}\right]^{v_2^*} w^{*a_1 v_1^* + a_2 v_2^*}, \end{aligned}$$

and hence from (A.5) we have

$$C(w^*) = \left[\frac{c_1}{v_1^*}\right]^{v_1^*} \left[\frac{c_2}{v_2^*}\right]^{v_2^*}.$$

Substituting the solution for the v_j^* from (A.6) and simplifying yields

$$C(w^*) = (a_2 - a_1)\left[\left(\frac{c_1}{a_2}\right)^{a_2}\left(-\frac{c_2}{a_1}\right)^{-a_1}\right]^{\frac{1}{a_2 - a_1}}. \tag{A.8}$$

To obtain the value for w^*, observe from (A.2) that

$$C(w^*) = \frac{c_1 w^{*a_1}}{v_1^*} = \frac{c_2 w^{*a_2}}{v_2^*}. \tag{A.9}$$

Substituting the solution for the v_j^* from (A.6) into (A.9) and solving for w^* gives

$$w^* = \left[\frac{-a_1 c_1}{a_2 c_2}\right]^{\frac{1}{a_2 - a_1}}. \tag{A.10}$$

It now is straightforward to derive from (A.1), (A.9), and (A.6) the cost sensitivity relationship

$$\frac{C(w)}{C(w^*)} = \frac{1}{a_2 - a_1}\left[a_2\left(\frac{w}{w^*}\right)^{a_1} - a_1\left(\frac{w}{w^*}\right)^{a_2}\right]. \tag{A.11}$$

Hence the relative sensitivity of total costs to a non-optimal choice of w depends only upon the exponents a_j and the amount of deviation relative to w^*. Expressions for evaluating the sensitivity of total costs to misestimates in the cost coefficients c_j can be obtained by substituting (A.10) into (A.11). For specific instances of the general formula (A.11), see Geoffrion [37, p. 107], Hadley and Whitin [44, p. 36], and Wagner [88, p. 818].

Appendix B. Price-Sensitive Demands

In this Appendix we explore the relationship of the unit-cost-minimizing market area model, which assumes perfectly inelastic demands, to models with price-sensitive demands. First, a "typical" model with price-sensitive demands is defined, and we derive a standard equilibrium solution for this model. Then we show how price-sensitive demands may be incorporated into the unit-cost-minimizing model and compare the solution obtained from that model with the equilibrium solution. We shall restrict our investigation to the GOMA (0, 1, D, M) model since this model has the property that all boundary points are equidistant from the market center, and the complications entailed by non-equidistant boundary points in other models are avoided (see, e. g., [43,50,70,71]).

To introduce price-sensitive demand, we modify assumption (a) to

(a′) Demand is distributed uniformly over an infinite plain with density $a - bp$ per unit area, where $a, b > 0$ and p is the effective delivered price at the demand point.

The assumption of such a linear demand function is standard in models of equilibrium spatial pricing.

From the various forms of spatial competition that have been considered, we shall examine the one postulated by Lösch [16,18,25,60,65,67,68]. "Löschian" spatial equilibrium requires assumption (f) to be altered to:

(f′) Subject to assumptions (a′) – (e′), each market area consists of a single firm that sets price so as to maximize profits. Free competitive entry of new firms then squeezes the market area for each firm to the minimum size that will yield non-negative profits.

We shall examine two common pricing policies: *mill pricing*, in which the firm charges a price p_m at its location of production and consumers pay for transportation; and *uniform delivered pricing*, in which the firm absorbs transportation costs and charges a delivered price p_d to all consumers.

For convenience, here we define R as the distance from the center to a vertex of the diamond-shaped market area; thus the area of the market is $2R^2$ and, with distance measured by the "Manhattan metric," *all* points on the perimeter of the

market are at distance R from the center. For $\beta = 1$ and Manhattan metric distance, the total quantity sold under mill pricing is

$$w_m = 4\int_0^R \int_0^{R-u} [a - b(p_m + t(u+v))]dvdu,$$

which yields

$$w_m = 2R^2\left[a - bp_m - \frac{2}{3}btR\right]. \tag{B.1}$$

Firm total profits are given by

$$Y(p_m, R) = (p_m - c)w_m - k. \tag{B.2}$$

Substituting (B.1) into (B.2) and setting

$$\frac{\partial Y(p_m^*, R)}{\partial p_m} = 0$$

yields the profit-maximizing mill price

$$p_m^* = \frac{1}{2}\left[\frac{a}{b} + c - \frac{2}{3}tR\right]. \tag{B.3}$$

Substituting (B.3) into (B.2) gives the maximum firm profits

$$Y(p_m^*, R) = \frac{R^2}{2b}\left(a - bc - \frac{2}{3}btR\right)^2 - k. \tag{B.4}$$

Competition drives $Y(p_m^*, R)$ to zero, and the equilibrium market radius R_e is determined from (B.4) by

$$\frac{R_e^2}{2b}\left(a - bc - \frac{2}{3}btR_e\right)^2 = k. \tag{B.5}$$

Since competition is claimed to produce the densest packing of firms subject to the non-negativity of firm profits, we define R_e as the smallest positive root of the quartic (B.5):

$$R_e = \left(\frac{3}{4bt}\right)\left[a - bc - \left((a-bc)^2 - \frac{8\sqrt{2}}{3}bt\sqrt{bk}\right)^{1/2}\right]. \tag{B.6}$$

Similar expressions have been derived by Mulligan [70] for various market shapes assuming Euclidean distance.

Under uniform delivered pricing, the total quantity sold by the firm is

$$w_d = 2R^2(a - bp_d). \tag{B.7}$$

To calculate firm profits, we must evaluate total transport costs:

$$4\int_0^R \int_0^{R-u} (a - bp_d)t(u+v)\,dv\,du = \frac{4}{3}t(a - bp_d)R^3.$$

Firm total profits are given by

$$Y(p_d, R) = (p_d - c)w_d - \frac{4}{3}t(a - bp_d)R^3 - k. \tag{B.8}$$

Substituting (B.7) into (B.8) and setting

$$\frac{\partial Y(p_d^*, R)}{\partial p_d} = 0$$

yields the profit-maximizing uniform delivered price

$$p_d^* = \frac{1}{2}\left[\frac{a}{b} + c + \frac{2}{3}tR\right]. \tag{B.9}$$

Substituting (B.9) into (B.8) yields the same expression for maximum firm profits given in (B.4), corresponding to a result derived by Beckmann and Ingene [7] and Beckmann [5]. Hence the equilibrium market area will be the same under either mill pricing or uniform delivered pricing, as observed by Mulligan [71] for several market shapes assuming Euclidean distance.

In order to introduce price-sensitive demands into the unit-cost-minimizing market area model, we shall adopt an institutional setting different from the one assumed in the above equilibrium models. There is substantial evidence that firms consider much more than short-run profit maximization in planning their investment, output, and pricing decisions. Weston [95] has mentioned in particular a concern with lowering costs and "enlarging the market." Firms are likely to adopt strategies to reduce the possibility that potential competitors will enter their market [27,51]. This suggests the firm objective we shall adopt here: to set an "entry-deterring" price at the lowest level possible, subject to a non-negativity constraint on profits. We emphasize that this analysis is an *ex ante* one intended to establish facility capacity and the structure of costs; actual *ex post* pricing and output decisions very likely could be adjusted according to shorter-run considerations.

We also shall adopt a strategy of uniform delivered pricing. Although this type of price policy is regarded as discriminatory, since prices do not reflect the costs of service, it is quite common among firms [74,76,78].

The problem of setting the entry-deterring price p, subject to non-negativity of firm profits $Y(p, w, D)$, is defined as

$$\begin{array}{rll} \text{Minimize} & p & \quad \text{(B.10)} \\ p, D \geq 0 & & \\ w > 0 & & \end{array}$$

$$\begin{array}{rll} \text{subject to} & Y(p, w, D) \geq 0 & \quad \text{(B.11)} \\ & D = a - bp. & \quad \text{(B.12)} \end{array}$$

Profits are defined as

$$Y(p, w, D) = (p - c - C(w, D))w$$

where $c + C(w, D)$ is the unit average total cost with $C(w, D)$ defined as in (3.2), introducing D as a variable.

Substituting from (B.12) for p into (B.10) and (B.11) yields the equivalent problem

$$\begin{array}{rll} \text{Maximize} & D & \quad \text{(B.13)} \\ a \geq D \geq 0 & & \\ w > 0 & & \end{array}$$

$$\text{subject to} \quad \left(\frac{a - D}{b} - c - C(w, D)\right) w \geq 0. \tag{B.14}$$

Observe that the constraint (B.14) must hold with equality at the optimum, since otherwise we would attempt to set $D = a$, which would cause the constraint to be violated. This implies that

$$D^* = a - b[c + C(w^*, D^*)] \tag{B.15}$$

which in turn requires that $C(w^*, D^*)$ be minimal with respect to w; i.e.

$$\frac{\partial C(w^*, D^*)}{\partial w} = 0. \tag{B.16}$$

Given D, the solution indicated by (B.16) may be obtained from (3.4). Also, from (B.12) and (B.15) we have

$$p^* = c + C(w^*, D^*). \tag{B.17}$$

The solution given here, as determined by (B.15) and (B.16), maximizes unit demand D and minimizes unit cost $C(w^*, D^*)$. This form of solution corresponds to the statement by Beckmann [2, p. 45] that for a uniform delivered price "The social optimum is then achieved by ... the minimization of unit cost." However, Beckmann did not indicate how the level of demand was to be chosen for the minimization of cost.

Analytical solution of (B.15) and (B.16) for w^* and D^* does not seem possible, but the following successive approximations approach converges monotonically and usually rapidly to the largest solution $D^* > 0$, if such a solution exists:

1. Set $D_0 = a - bc$; $\ell = 0$.

2. $D_{\ell+1} = a - b[c + C(w^*, D_\ell)]$.

3. If $D_{\ell+1}$ has converged to D^*, stop.
   Otherwise $\ell \leftarrow \ell + 1$ and return to Step 2.

The solution $C(w^*, D_\ell)$ in Step 2 is obtained from (3.4). Monotonicity of convergence results from $D^* \leq D_0 = a - bc$; the observation that $C(w^*, D)$ is strictly decreasing in D in (3.4); and the strictly decreasing nature of $D = a - b[c + C(w^*, D)]$ in $C(w^*, D)$. A second-order approach, such as Newton's method, would provide even more rapid convergence.

To illustrate the solution process for price-sensitive demands, we will calculate a solution for GOMA (0, 1, D, M) using the following data from Hartwick [50] and Mulligan [70]: $a = 25$, $b = 5$, $t = 1$, $c = 0.6$, $k = 5$. For this model, we have from (3.3)

$$w^* = \left[\frac{2kD^{1/2}}{t\sigma(1,\ \mathrm{D},\ \mathrm{M})}\right]^{2/3}$$

and from (3.4)

$$C(w^*, D) = \left(\frac{3}{2}\right)\left[(2k)^{1/2} t\sigma(1,\ \mathrm{D},\ \mathrm{M}) D^{-1/2}\right]^{2/3}$$

where $\sigma(1,\ \mathrm{D},\ \mathrm{M}) = \sqrt{2}/3$ from Table 1. Results from the successive approximations approach are as follows:

ℓ	Price, p_ℓ	Demand, D_ℓ	Cost, $C(w_\ell, D_\ell)$	Size, w_ℓ
0	0.600	22.00	0.6986	21.47
1	1.299	18.51	0.7400	20.27
2	1.340	18.30	0.7428	20.19
3	1.343	18.29	0.7430	20.19

The optimal market radius $R^* = \sqrt{w^*/2D^*} = C(w^*, D^*)/t = 0.7430$.

For contrast, the solution for the equilibrium model yields via (B.6) an equilibrium market radius R_e of 0.3388. This implies a uniform delivered price of 2.913, a demand density D of 10.44, an average unit cost (excluding the proportional amount c) of 2.313, and a facility size of 2.396. Clearly such small, high-cost producers would stand little competitive chance against the large, efficient firm generated by the GOMA (0, 1, D, M) model. Even with a relatively insensitive cost model, facility size differentials that approach an order of magnitude make a noticeable difference!

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